# A STUDY SLIP FLOW THROUGH ELLIPTIC MICROCHANNELS

**S. Suharsono**
Department of Mathematics, Faculty of Mathematics and Natural Sciences,
Universitas Lampung, Lampung 35145, Indonesia
email: suharsono.1962@fmipa.unila.ac.id

**ABSTRACT:** *There are many publications explored steady slip flow through micro channel such as micro annuals, rectangular, elliptical. This study solves analytically unsteady slip flow through elliptic micro channels in case of constant pressure gradient. The exact solutions for the velocity field were found by variable separable method.*

**Keywords:** *Unsteady, Slip flow, Elliptic micro-channel, constant pressure gradient.*

## 1. INTRODUCTION

In the past decade, there are various applications of fluid flow in microchannels such as in industrial, computer chips and chemical processing, etc. It has emerged an important research area. Understanding the profile of fluid flow in microchannels is very important to determine velocity profile, pressure distribution and properties of the flow. It happens after finding analitical or numerical solutions of the flow.

Many publications studied fluid flow through micro-channels in many cross-sections, such as trapezoidal, annulus, rectangular and elliptical. Some of them completed by no-slip boundary conditions and the other by slip boundary conditions. There are one for steady case and the other for unsteady case, analytically or experimentally. Most of them studied for rectangular and circular cross-sections [ 1, 2, 3, 4, 5, 6, 7 ]. In elliptic cross section, Samir and Farzad [11] investigate fully developed laminar hydro-dynamically steady state and incompressible with constant fluid properties. Recently, Chuchard et al [8] analytically studied an unsteady electroosmotic and pulsatile flow through an elliptic cylindrical microchannel with the Navier slip boundary. Previously, Duan and Muzychka [10] studied an exact solution of a steady slip flow of Newtonian fluid for constant pressure gradient in elliptic micro-channels.

The objective of this research is to derive an analytical solution of transient flow of a Newtonian fluid with constant pressure gradient in elliptic micro-channel with Navier slip boundary. This paper organizes as follows. In section 2, the Navier Stokes equation in rectangular coordinates is transformed to elliptic cylindrical coordinates. In section 3, the derivation of an analytically is given.

## 2. GOVERNING EQUATIONS

Consider the unsteady Navier-Stokes equation in rectangular coordinates which is derived in paper [9]:

$$\frac{\mu}{\rho}\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)-\frac{\partial u}{\partial t}=\frac{1}{\rho}\frac{dp}{dz} \qquad (1)$$

We transform rectangular coordinates $(x,y,z)$ to elliptic cylindrical coordinates $(\eta,\psi,z)$ using the following coordinate transformation

$$x=c\cosh\eta\cos\psi$$
$$y=c\sinh\eta\sin\psi;\ \ 0\le\eta<\infty,\ 0\le\psi<2\pi \qquad (2)$$
$$z=z;\quad -\infty<z<+\infty.$$

The metric coefficients are defined by

$$g_{ii}=\left(\frac{\partial x^1}{\partial u^i}\right)^2+\left(\frac{\partial x^2}{\partial u^i}\right)^2+\left(\frac{\partial x^3}{\partial u^i}\right)^2 \qquad (3)$$

so that $g_{11}=g_{22}=c^2\left(\cosh^2\eta-\cos^2\psi\right)$ and

$$g=\left(g_{11}.g_{22}.g_{33}\right)=c^4\left(\cosh^2\eta-\cos^2\psi\right)^2, \qquad (4)$$

The Laplacian of *v* in elliptic cylinder coordinates is defined by

$$\nabla^2 v=g^{1/2}\sum_{i=1}^{3}\frac{\partial}{\partial u^i}\left(\frac{g^{1/2}}{g_{ii}}\frac{\partial v}{\partial u^i}\right);\quad \left(u^1,u^2,u^3\right)=\left(\eta,\psi,z\right). \qquad (5)$$

Thus

$$\nabla^2 v=\frac{1}{c^2\left(\cosh^2\eta-\cos^2\psi\right)}\left(\frac{\partial}{\partial\eta}\left(\frac{g^{1/2}}{g_{11}}\frac{\partial v}{\partial\eta}\right)+\frac{\partial}{\partial\psi}\left(\frac{g^{1/2}}{g_{22}}\frac{\partial v}{\partial\psi}\right)+\frac{\partial}{\partial z}\left(\frac{g^{1/2}}{g_{33}}\frac{\partial v}{\partial z}\right)\right) \qquad (6)$$
$$=\frac{1}{c^2\left(\cosh^2\eta-\cos^2\psi\right)}\left(\frac{\partial^2 v}{\partial\eta^2}+\frac{\partial^2 v}{\partial\psi^2}\right)+\frac{\partial^2 v}{\partial z^2}.$$

As there is no swirling flow, $\frac{\partial v}{\partial z}=0$. Thus the momentum equation (1) written in elliptic cylindrical coordinates is

$$\frac{1}{c^2\left(\cosh^2\eta-\cos^2\psi\right)}\left(\frac{\partial^2 v}{\partial\eta^2}+\frac{\partial^2 v}{\partial\psi^2}\right)-\frac{\rho}{\mu}\frac{\partial v}{\partial t}=\frac{1}{\mu}\frac{dp}{dz}. \qquad (7)$$

Assuming the boundary conditions in a one quarter basic cell, are

(i) $\frac{1}{\sqrt{g_{22}}}\frac{\partial v}{\partial\psi}(\eta,0)=0$ ,

(ii) $\frac{1}{\sqrt{g_{22}}}\frac{\partial v}{\partial\psi}\left(\eta,\frac{\pi}{2}\right)=0$, (8)

(iii) $\frac{1}{\sqrt{g_{11}}}\frac{\partial v}{\partial\eta}(0,\psi)=0$, and

(iv) $v+\frac{\ell}{\sqrt{g_{11}}}\frac{\partial v}{\partial\eta}(\eta_0,\psi)=0;\quad \eta_0=\ln\frac{1+b/a}{\sqrt{1-(b/a)^2}};$

$$c=\frac{a}{\cosh\eta_0}=\frac{b}{\sinh\eta_0},$$

where $\ell$ is slip length, which is defined by $\ell = \lambda \frac{2-\sigma}{\sigma}$, where $\lambda$ is the molecular mean free path and $\sigma$ denotes the tangential momentum accommodation coefficient, which has values between 0.87 and 1. In case of $\ell = 0$, conditions (8) reduces to the no-slip boundary condition.

**3. SOLUTION OF VELOCITY FIELD**

We let $\frac{dp}{\partial z} = c_0$ and apply equality

$$\cosh^2\eta - \cos^2\psi = (\cosh 2\eta - \cos 2\psi)/2 \quad (9)$$

such that we have

$$\frac{2}{c^2(\cosh 2\eta - \cos 2\psi)}\left(\frac{\partial^2 v}{\partial \eta^2} + \frac{\partial^2 v}{\partial \psi^2}\right) - \frac{\rho}{\mu}\frac{\partial v}{\partial t} = \frac{c_0}{\mu}. \quad (10)$$

Then let $v(\eta,\psi,t) = w(\eta,\psi) + u(\eta,\psi,t)$, so that

$$v_{\eta\eta} = w_{\eta\eta} + u_{\eta\eta}$$
$$v_{\psi\psi} = w_{\psi\psi} + u_{\psi\psi} \quad (11)$$
$$v_t = u_t.$$

Therefore we have

$$\frac{2}{c^2(\cosh 2\eta - \cos 2\psi)}\left(w_{\eta\eta} + u_{\eta\eta} + w_{\psi\psi} + u_{\psi\psi}\right) - \frac{\rho}{\mu}u_t = \frac{c_0}{\mu} \quad (12)$$

which is equivalent to

$$\frac{2}{c^2(\cosh 2\eta - \cos 2\psi)}\left(u_{\eta\eta} + u_{\psi\psi}\right) - \frac{\rho}{\mu}u_t = 0 \quad (13)$$ and

$$\frac{2}{c^2(\cosh 2\eta - \cos 2\psi)}\left(w_{\eta\eta} + w_{\psi\psi}\right) = \frac{c_0}{\mu} \quad (14)$$

To solve eq (13), we let $u(\eta,\psi,t) = u^{(1)}(\eta,t) + u^{(2)}(\psi,t)$ so that we have

$$\left(u^{(1)}_{\eta\eta} + u^{(2)}_{\psi\psi}\right) = \frac{\rho c^2}{2\mu}u_t(\cosh 2\eta - \cos 2\psi) \quad (15)$$

which imply

$$u^{(1)}_{\eta\eta} = a_1 u^{(1)}_t \cosh 2\eta \quad (16)$$

and

$$u^{(2)}_{\psi\psi} = -a_1 u^{(2)}_t \cos 2\psi \quad \text{where} \quad a_1 = \frac{\rho c^2}{2\mu}. \quad (17)$$

Applying separation variable methods to solve eq (15) and (16) by letting

$$u^{(1)}(\eta,t) = F(\eta)G(t) \quad (18)$$

and

$$u^{(2)}(\psi,t) = P(\psi)Q(t) \quad (19)$$

such that we have

$$(i) \quad F_{\eta\eta} - k_1^2 F\cosh(2\eta) = 0$$
$$(ii) \quad G_t + \frac{k_1^2}{a_1}G = 0 \quad (20)$$

and

$$(i) \quad P_{\psi\psi} + k_2^2 P\cos(2\psi) = 0$$
$$(ii) \quad Q_t - \frac{k_2^2}{a_2}Q = 0$$

(21)

Eq (20)(i) and (21)(i) are Mathieu equations which having solutions

$$F(\eta) = C_1 Ce_{2n}\left(\eta, -\frac{1}{2}k_1^2\right) + C_2 Se_{2n}\left(\eta, -\frac{1}{2}k_1^2\right) \quad (22)$$

and

$$P(\psi) = C_3 ce_{2n}\left(\psi, -\frac{1}{2}k_1^2\right) + C_4 se_{2n}\left(\psi, -\frac{1}{2}k_1^2\right). \quad (23)$$

Eq (20)(ii) and (21)(ii) are respectively having solution

$$G(t) = A\exp(-\frac{k_1^2}{a_1}t) \quad (24)$$

and

$$Q(t) = B\exp(\frac{k_2^2}{a_1}t) \quad (25)$$

To solve eq (14) we first write it becomes

$$\left(w_{\eta\eta} + w_{\psi\psi}\right) = \frac{c_0 c^2}{2\mu}(\cosh 2\eta - \cos 2\psi) \quad (26)$$

which is equivalent to

$$w_{\eta\eta} - b_1\cosh 2\eta = 0 \quad (27)$$

and

$$w_{\psi\psi} + b_1\cos 2\psi = 0 \quad (28)$$

where $b_1 = \frac{c_0 c^2}{2\mu}$ .

Integrate (27) twice so that

$$w(\eta,\psi) = \frac{b_1}{4}\cosh 2\eta + \eta f(\psi). \quad (29)$$

Then, differentiate *w* over $\psi$, and applying eq (28) which yield

$f''(\psi) = -\frac{b_1}{\eta}\cos 2\psi$ so that $f(\psi) = -\frac{b_1}{4\eta}\cos 2\psi$.

Hence $w(\eta,\psi) = \frac{b_1}{4}\cosh 2\eta + \frac{b_1}{4\eta}\cos 2\psi$. (30)

Now applying boundary conditions (BC) (8) (i) – (iv).

As $v(\eta,\psi,t) = w(\eta,\psi) + u(\eta,\psi,t)$

$$= w(\eta,\psi) + u^{(1)}(\eta,t) + u^{(2)}(\psi,t), \quad (31)$$

Applying BC(8)(i) $\frac{\partial v}{\partial \psi}(\eta,0) = 0$ implies that

$w_\psi(\eta,0) + u^{(2)}_\psi(0,t) = 0$, then we have $C_4 = 0$.

Using BC (8)(iii), $w_\eta(0,\psi) + u^{(1)}_\eta(0,t) = 0$ which gives $C_2 = 0$. Therefore

$$v_n(\eta,\psi,t)=w_n(\eta,\psi)+u_n(\eta,\psi,t)$$
$$=\frac{b_1}{4}\left(\cosh 2\eta+\frac{1}{\eta}\cos 2\psi\right)$$
$$+A_nCe_{2n}\left(\eta,-\frac{k_1^2}{2}\right)e^{-\frac{k_1^2t}{a_1}}$$
$$+B_nce_{2n}\left(\psi,-\frac{k_2^2}{2}\right)e^{-\frac{k_2^2t}{a_1}}. \quad (32)$$

Boundary condition (8)(ii) automatically satisfied.

To apply BC (8)(iv), write

$$\frac{\partial v_n}{\partial\eta}(\eta_0,\psi,t)=\frac{b_1}{2}\sinh 2\eta_0$$
$$+A_n\left(\sum_{r=1}^{\infty}2rA_{2r}^{2n}\sinh(2r\eta_0)\right)e^{-\frac{k_1^2t}{a_1}}$$
$$=\frac{b_1}{2}\sinh 2\eta_0+A_nh_n(\eta_0)e^{-\frac{k_1^2t}{a_1}}, \quad (33)$$

where $h_n(\eta_0)=\sum_{r=1}^{\infty}2rA_{2r}^{2n}\sinh(2r\eta_0)$.

$$v_n(\eta_0,\psi,t)=\frac{b_1}{4}\left(\cosh 2\eta_0+\frac{1}{\eta}\cos 2\psi\right)$$
$$+A_nCe_{2n}\left(\eta_0,-\frac{k_1^2}{2}\right)e^{-\frac{k_1^2t}{a_1}}$$
$$+B_nce_{2n}\left(\psi,-\frac{k_2^2}{2}\right)e^{-\frac{k_2^2t}{a_1}}$$
$$=g(\psi)+A_nCe_{2n}\left(\eta_0,-\frac{k_1^2}{2}\right)e^{-\frac{k_1^2t}{a_1}}$$
$$+B_nce_{2n}\left(\psi,-\frac{k_2^2}{2}\right)e^{-\frac{k_2^2t}{a_1}} \quad (34)$$

.

Applying BC (8)(iv),

$$v_n+\frac{\ell}{\sqrt{g_{11}}}\frac{\partial v_n}{\partial\eta}(\eta_0,\psi)=0\text{, ie}$$

$$g(\psi)+A_nCe_{2n}\left(\eta_0,-\frac{k_1^2}{2}\right)e^{-\frac{k_1^2t}{a_1}}$$
$$+B_nce_{2n}\left(\psi,-\frac{k_2^2}{2}\right)e^{-\frac{k_2^2t}{a_1}}$$
$$+\frac{\ell}{a}f(\psi)\left(\frac{b_1}{2}\sinh 2\eta_0+A_nh_n(\eta_0)e^{-\frac{k_1^2t}{a_1}}\right)=0. \quad (35)$$

where

$$\frac{\ell}{\sqrt{g_{11}}}=\frac{\ell}{a}\left(1-\frac{1}{2}\frac{\cos^2\psi}{\cosh^2\eta_0}-\frac{1}{8}\frac{\cos^2\psi}{\cosh^2\eta_0}\right)=\frac{\ell}{a}f(\psi)$$

Because this equation holds for any instant of time $t$, then for $t=0$ yields

$$g(\psi)+A_nCe_{2n}\left(\eta_0,-\frac{k_1^2}{2}\right)+B_nce_{2n}\left(\psi,-\frac{k_2^2}{2}\right)+$$
$$\frac{\ell}{a}f(\psi)\left(\frac{b_1}{2}\sinh 2\eta_0+A_nh_n(\eta_0)\right)=0.$$

$$\leftrightarrow\left[g(\psi)+B_nce_{2n}\left(\psi,-\frac{k_2^2}{2}\right)\right]$$
$$+\left[A_n\left(Ce_{2n}\left(\eta_0,-\frac{k_1^2}{2}\right)+\frac{\ell}{a}f(\psi)h_n(\eta_0)\right)\right.$$
$$\left.+\frac{\ell b_1}{2a}f(\psi)\sinh 2\eta_0\right]=0 \quad (36)$$

which every term can be choosen to be 0.
Multiply the first term in the above equation with $ce_{2n}\left(\psi,-\frac{k_2^2}{2}\right)$ and apply the ortogonality principle, then we have

$$B_n=\int_0^{2\pi}-g(\psi)\,ce_{2n}\left(\psi,-\frac{k_2^2}{2}\right)d\psi$$
$$\times\left(\int_0^{2\pi}ce_{2n}^2\left(\psi,-\frac{k_2^2}{2}\right)d\psi\right)^{-1}. \quad (37)$$

For the second term

$$A_n=\int_0^{2\pi}\frac{\ell b_1}{2a}f(\psi)\sinh 2\eta_0\,d\psi$$
$$\times\left(\int_0^{2\pi}Ce_{2n}\left(\eta_0,-\frac{k_1^2}{2}\right)+\frac{\ell}{a}f(\psi)h_n(\eta_0)d\psi\right)^{-1} \quad (38)$$

Therefore, general solution of eq. (1) has the form

$$v(\eta,\psi,t)=\sum_{n=1}^{\infty}v_n(\eta,\psi,t)$$
$$=\frac{b_1}{4}\left(\cosh 2\eta+\frac{1}{\eta}\cos 2\psi\right)$$
$$+\sum_{n=1}^{\infty}A_nCe_{2n}\left(\eta_0,-\frac{k_1^2}{2}\right)e^{-\frac{k_1^2t}{a_1}}$$
$$+\sum_{n=1}^{\infty}B_nce_{2n}\left(\psi,-\frac{k_2^2}{2}\right)e^{-\frac{k_2^2t}{a_1}} \quad (39)$$

where $A_n$ and $B_n$ are defined as above.

## 4. CONCLUSION

The governing equation (1) completed by boundary conditions in elliptic micro-channel. The separation method was applied to solve eq (1) for the case of constant pressure gradient by letting $v(\eta,\psi,t)=w(\eta,\psi)+u(\eta,\psi,t)$ $=w(\eta,\psi)+u^{(1)}(\eta,t)+u^{(2)}(\psi,t)$ and yields an exact solution for the velocity field.

## ACKNOWLEDGEMENTS

**T**he author gratefully acknowledges to Professor Mustofa Usman for useful suggestions.

## REFERENCES

[1] Z. X. Li, D. X. Du, and Z. Y. Guo. Experiment study on ow characteristics of liquid in circular microtubes, *Proceeding of the International Conference on Heat Transfer and Transport Phenomena in Microscale*, Ban_, Canada, pp.
162-167, October 15-20, (2000).

[2] M. J. Kohl, S. I. Abdel-Khalik, S. M. Jeter, and D.L. Sadowski. An experimental investigation of microchannel ow with internal pressure measurements, *International Journal of Heat and Mass Transfer*, 48, 1518-1533 (2005)

[3] B. Wiwatanapataphee, Y.H. Wu, H. Maobin and K. Chayantrakom. A study of transient Flows of Newtonian Fluid through micro-annuals withslip boundary, *J. Phys. A:Math. Theor*. 42, 065206 (2009).

[4] R. Baviere, F. Ayela, S. Le Person, and M. Farve-Marient. An experiment study of water flow in smooth and rough rectangular microchannels,*Paper No.ICMM2004-2338, Second Inter- national Conference on Microchannels and Minichannels*, Rochester, NY USA, pp. 221-228,June 17-19,( 2004).

[5] D. Lelea, S. Nishio, and K. Takano. The experimental research on microtube heat transfer and fluid flow of distilled water, *International Journal of Heat and Mass Transfer*, 47, 2817-2830 (2004)

[6] Y.H. Wu, B. Wiwatanapataphee and H. Maobin. Pressure-dricen transient flows of Newtonian fluids through microtubes with slip boundary,*Physica A*, 387, 5979-5990 (2008).

[7] G. H. Tang, Z. Li, Y. L. He, and W. Q. Tao. Experimental study of compressibility, roughness and rarefaction influences on microchannelflow, *International Journal of Heat and Mass Transfer*, 50, 2282-2295 (2007).

[8] P. Chuchard, Somsak O and B, Wiwatanapataphee. tudy of pulsatile pressure driven electroosmotic flows through an alliptic cylindrical microchannel with the Navier slip condition. *Advances in Difference Equations. 2017:160* (2017)

[9] B. Wiwatanapataphe, Yong Hong Wu, Suharsono Suharsono. Transient Flows of Newtonian Fluid through a Rectangular Microchannel with Slip Boundary. *Abstract and Applied Analysis*. Volume 2014, Article ID 530605, 13 pages.

[10] Z. Duan, Y.S. Muzychka, *Slip flow in ellipticmicro-channels*, Int.Journal of Thermal Sciences 46. 1104–1111 (2007).

[11] Samir K Das, F. Tahmouresi, Analitical solution of fully developed gaseous slip flow in elliptic microchannel. *Int. J. Adv. Appl. Math. And Mech*. 3 (3), (2016) 1 – 15 (ISSN: 2347 – 2529 )